1. Lusin's Theorem and applications

Egoroff's Theorem says that on a set of finite measure, almost everywhere convergence of measurable functions to a finite limit is uniform convergence off of a set of small measure. A consequence (see Problem 31 on Page 74) is Lusin's Theorem, which says that on a set of finite measure, any finite measurable function f can be restricted to a compact set K of almost full measure to form a continuous function. We will present a new simple proof of Lusin's theorem due to Erik Talvila and P. Loeb. Let us recall the following result proved in a homework:

LEMMA 1.1. *Given a measurable set $A \subseteq \mathbb{R}$ with $m(A) < +\infty$, and given $\varepsilon > 0$, there is a compact set $K \subseteq A$ with $m(A \setminus K) < \varepsilon$.*

PROOF. We already know that there is a closed subset F of A with $m(A \setminus F) < \varepsilon/2$. Since the sequence

$$F \cap [-n, n] \nearrow F,$$

and $m(F) < +\infty$, there is an n_0 such that $m(F \setminus [-n_0, n_0]) < \varepsilon/2$. The desired compact set is $F \cap [-n_0, n_0]$. $\square$

THEOREM 1.2 (Lusin). *Fix a measurable set $A \subseteq \mathbb{R}$ with $m(A) < +\infty$, and let f be a real-valued measurable function with domain A. For any $\varepsilon > 0$, there is a compact set $K \subseteq A$ with $m(A \setminus K) < \varepsilon$ such that the restriction of f to K is continuous.*

PROOF. Let $\langle V_n \rangle$ be an enumeration of the open intervals with rational endpoints in $\mathbb{R}$. Fix compact sets $K_n \subseteq f^{-1}[V_n]$ and $K_n' \subseteq A \setminus f^{-1}[V_n]$ for each n so that $m(A \setminus (K_n \cup K_n')) < \varepsilon/2^n$. Now, for $K := \bigcap_n (K_n \cup K_n')$, $m(A \setminus K) < \varepsilon$. Given $x \in K$ and an n with $f(x) \in V_n$, $x \in O := \widetilde{K_n'}$ and $f\,[O \cap K] \subseteq V_n$. ■

This result is true in quite general settings. In the general setting, you may see this result stated just for Borel measurable functions. The domain of f should have the property that sets of finite measure can be approximated from the inside by compact sets, and for the range, there should be a countable collection of open sets V_n such that for each open set O and each $y \in O$ there is an n with $y \in V_n \subseteq O$. (This is called the Second Axiom of Countability.)

COROLLARY 1.3. *Let A be a measurable set such that $m(A) < \infty$. Let $f : A \to \mathbb{R}$ be measurable function and $\varepsilon > 0$. Then there exists a step function $h : \mathbb{R} \to \mathbb{R}$ such that*

$$m(|f - h| \geq \varepsilon) < \varepsilon\,.$$

Moreover, if f is bounded than

$$\sup|h| \;\leq\; \sup|f|\;.$$

PROOF. Let K be such that $f|K$ is continuous and

$$m(A\mathcal{K}) < \varepsilon\;.$$

Being compact we known that K is bounded , say $K \subset [-N,N]$. Since $f_{|}K$ is continuous it is also uniform continuous. Thus we may find $0<\delta<\varepsilon$ such that

$$t,s\in K \text{and} |t-s|<\delta \Longrightarrow |f(t)-f(s)|<\varepsilon\;.$$

Let $n>\delta^{-1}$ and $x_i = -N+\frac{i}{n}$, $i=0,....,2Nn$. Let S be the collection of indices such that there exists $i\in K$ such that $[x_i,x_{i+1})\cap K\neq\emptyset$. For such $i\in S$ we may pick $y_i\in[x_i,x_{i+1}]$. We define the step function

$$h \;=\; \sum_{i\in S} f(y_i)1_{[x_i,x_{i+1})}\;.$$

Let $s\in K$. Choose $i=0,....,2Nm$ such that $x_i\leq s<x_{i+1}$. Then $K\cap[x_i,x_{i+1})\cap K\neq\emptyset$ and $|y_i-s|<\frac{1}{n}<\delta$. We get

$$|h(s)-f(s)| \;=\; |f(y_i)-f(s)|<\varepsilon\;.$$

Thus

$$m(|h-f|\geq\varepsilon) \;\leq\; m(A\setminus K)<\varepsilon\;.$$

Since h is constructed using the elements $f(y_i)$ we also get

$$\sup_{x\in\mathbb{R}}|h(x)| \;\leq\; \sup_{x\in K}|f(x)|\;.$$

This implies the second assertion. ■

COROLLARY 1.4. *Let $A\subset\mathbb{R}$ be a measurable set and $f:A\to\mathbb{R}$ be a measurable function and $\varepsilon>0$. Then there exists a continuous function h such that*

$$m(|f-h|>\varepsilon)<\varepsilon\;.$$

Moreover, we can choose h such that

$$\sup|h| \;\leq\; \sup|f|pl.$$

PROOF. It suffices to show that for every simple function $f = \sum_{i=1}^{m} r_i 1_{[x_i,x_{i+1})}$ we can find a continuous h with

$$\mu(|f-h|>\varepsilon)<\varepsilon \quad \text{and} \quad |h|\le |f|\ .$$

It is easily shown by induction that

$$\mu(|(\sum_i f_i)-(\sum_i h_i)|>\sum_i \varepsilon_i) \ \le \ \sum_i \mu(|f_i-h_i|>\varepsilon_i)\ .$$

Therefore it suffices to consider $f_i \ = \ 1_{[x_i,x_{i+1})}$. Let $0<2\delta<x_{i+1}-x_i$ we define

$$h_{i,\delta}(t) \ = \ \begin{cases} \delta^{-1}(t-x_i) & \text{if } x_i<t\le x_i+\delta\ , \\ 1 & \text{if } x_i+\delta\le t\le x_{i+1}-\delta\ , \\ \delta^{-1}(x_{i+1}-t) & \text{if } x_{i+1}-\delta\le t\le x_{i+1} \\ 0 & \text{else} \end{cases} .$$

Note that $h_{i,\delta}\le 1_{[x_i,x_{i+1})}$ is continuous and that

$$m(|h_{i,\delta}-1_{[x_i,x_{i+1})}|>0)<2\delta\ .$$

Let δ such that $\frac{2\delta}{m}<\min_i(x_{i+1}-x_i)$. Then we may define

$$h \ = \ \sum_i r_i h_{i,\frac{\delta}{m}}$$

Then we have

$$m(|f-h|>\delta) \ \le \ \sum_{i=1}^{m} m(r_i|1_{[x_i,x_{i+1})}-h_{i,\frac{\delta}{m}}|>\frac{\delta}{m})<2m\frac{\delta}{m}<2\delta\ .$$

For the second assertion, we note that

$$|h|\le |f|\ .$$

Therefore we also control the sup-norm. ■